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Why Modulate?

Digital modulation
- the process by which digital symbols are transformed into waveforms that are compatible with characteristics of the channel

Impulse Sampling

Why use a carrier?
- Reduce size of antenna
  - Ex) 3000Hz baseband signal

\[
\lambda = \frac{c}{f} = \frac{3 \times 10^8 \text{m/sec}}{f}
\]

\[
\Rightarrow \quad \frac{\lambda}{4} = 2.5 \times 10^4 \text{m} = 8 \text{cm}
\]

- Frequency division multiplexing
- Minimize the effects of interference → spread spectrum
Digital Bandpass Modulation Techniques (1)

- **Bandpass modulation**
  - The process by which an information signal is converted to a sinusoidal waveform

- **General form of the carrier waveform**
  - $s(t) = A(t)\cos[\omega_0 t + \Phi(t)]$
  - $A(t) \rightarrow ASK, \omega_0 \rightarrow FSK, \Phi(t) \rightarrow PSK$

- **Basic bandpass modulation / demodulation types**

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<td><strong>Coherent</strong></td>
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<td>Phase shift keying (PSK)</td>
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<tr>
<td>Hybrids</td>
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</table>
Digital Bandpass Modulation Techniques (2)

- Coherent demodulation
  - receiver exploits knowledge of carrier's phase to detect the signal
  - operate with knowledge of the absolute value of incoming signal's phase

- Noncoherent demodulation
  - receiver does not utilize phase reference information
  - operate without knowledge of the absolute value of incoming signals phase
  - phase estimation is not required
  - reduced complexity
  - increased Probability of error ($P_E$)
Digital Bandpass Modulation Techniques (3)

- Phasor Representation of a Sinusoid
  - Euler's theorem
    - $e^{j\omega_0 t} = \cos(\omega_0 t) + jsin(\omega_0 t)$
Phasor Representation of a Sinusoid

**AM (amplitude modulation)**

\[ s(t) = \text{Re} \left\{ \left( 1 + \frac{e^{j\omega_m t} + e^{-j\omega_m t}}{2} \right) e^{j\omega_0 t} \right\} \]

**FM (frequency modulation)**

\[ s(t) = \text{Re} \left\{ e^{j\omega_0 t} \left( 1 - \frac{\beta}{2} e^{-j\omega_m t} + \frac{\beta}{2} e^{j\omega_m t} \right) \right\} \]
Waveform Amplitude Coefficient

\[ s(t) = A \cos(\omega t) \]

\[
\left( \text{Arms} = \sqrt{\frac{1}{T} \int_{0}^{T} s^2(t) \, dt} = \frac{A}{\sqrt{2}} \rightarrow A = \sqrt{2} \text{Arms} \right)
\]

\[
= \sqrt{2A^2 \text{rms}} \cos(\omega t) = \sqrt{2P} \cos(\omega t) = \sqrt{\frac{2E}{T}} \cos(\omega t)
\]

\[(A^2 \text{rms} = P \text{ (average power)})\]
Phasor Representation of a Sinusoid (cont)

Example of the most common digital modulation

(a) PSK
\[ s_i(t) = \sqrt{\frac{2E}{T}} \cos (\omega_0 t + 2\pi i/M) \]
\[ i = 1, 2, \ldots, M \]
\[ 0 \leq t \leq T \]

(b) FSK
\[ s_i(t) = \sqrt{\frac{2E}{T}} \cos (\omega_i t + \phi) \]
\[ i = 1, 2, \ldots, M \]
\[ 0 \leq t \leq T \]

(c) ASK
\[ s_i(t) = \sqrt{\frac{2E_i(t)}{T}} \cos (\omega_0 t + \phi) \]
\[ i = 1, 2, \ldots, M \]
\[ 0 \leq t \leq T \]

(d) ASK/PSK (APK)
\[ s_i(t) = \sqrt{\frac{2E_i(t)}{T}} \cos [\omega_0 t + \phi_i(t)] \]
\[ i = 1, 2, \ldots, M \]
\[ 0 \leq t \leq T \]
Phase Shift Keying

- General analytic expression
  \[ s_i(t) = \sqrt{\frac{2E}{T}} \cos[\omega_0 t + \Phi_i(t)] , \quad \begin{cases} 0 \leq t \leq T \\ i = 1, 2, \ldots, M \end{cases} \]

- Binary PSK (BPSK) M=2
  - modulation data signal shifts the phase of the waveform \( s_i(t) \) to one of two states, either zero or \( \pi(180^\circ) \)
  - abrupt phase changes at the symbol transition \( \Rightarrow \) antipodal signal sets

Frequency Shift Keying

- General analytic expression
  \[ s_i(t) = \sqrt{\frac{2E}{T}} \cos[\omega_i t + \Phi_i] , \quad \begin{cases} 0 \leq t \leq T \\ i = 1, 2, \ldots, M \end{cases} \]

- FSK waveform
  - signal sets can be characterized with mutually perpendicular vectors (mutually perpendicular axes represents a sinusoid with a different frequency) \( \Rightarrow \) orthogonal signals
Digital Bandpass Modulation Techniques (8)

- **Amplitude Shift Keying**
  - General analytic expression
    
    \[
    s_i(t) = \sqrt{\frac{2E_i(t)}{T}} \cos[\omega_0 t + \Phi] , \quad \begin{cases} 
    0 \leq t \leq T \\
    i = 1, 2, \ldots, M
    \end{cases}
    \]
  - Binary ASK (on-off keying)

- **Amplitude Phase Keying**
  - General analytic expression
    
    \[
    s_i(t) = \sqrt{\frac{2E_i(t)}{T}} \cos[\omega_0 t + \Phi_i(t)] , \quad \begin{cases} 
    0 \leq t \leq T \\
    i = 1, 2, \ldots, M
    \end{cases}
    \]
  - combination of ASK and PSK
  - simultaneous phase and amplitude at the symbol transition time

- **QAM (Quadrature Amplitude Modulation)**
  - the set of M symbols in the two-dimensional signal space are arranged in a rectangular constellation
Detection of Signal in Gaussian Noise (1)

Decision Regions

Consider: two-dimensional signal space

\[ M = 2 \]

Detector's task

decide which of the signals was transmitted

→ "Minimum - error decision rule"
Detection of Signal in Gaussian Noise (2)

- Using Euclidean Distance: choose the signal class that the distance
  \[ d(r, s_i) = \| \hat{r} - \bar{s}_i \| \] is minimized

- Using decision region
  - received signal \( \hat{r} \) is located in region 1 \( \rightarrow \) choose signal \( \bar{s}_1 \)
  - received signal \( \hat{r} \) is located in region 2 \( \rightarrow \) choose signal \( \bar{s}_2 \)
Detection of Signal in Gaussian Noise (3)

Correlation Receiver

Consider: detection of bandpass signals

Received signal

\[ r(t) = s_i(t) + n(t) \]

\[ \begin{align*}
&0 \leq t \leq T \\
&i = 1, 2, \ldots, M \\
&s_i(t): \text{transmitted prototype signal} \\
&n(t): \text{random noise}
\end{align*} \]

Detection Process

step1 ⇒ "waveform-to-random-variable transformation" by using MF or correlator

⇒ transforming the waveform into a point in the decision space

step2 ⇒ "symbol decision" by comparing \( z(T) \) to a threshold or by choosing the max \( z_1(T) \) determine in which decision region the point is located
Detection of Signal in Gaussian Noise (4)

- Correlation Receiver (cont)
- Detection Process (cont)
Correlator receiver with reference signals \{s_i(t)\}

- Transform a received waveform \(r(t)\) to a sequence of \(M\) correlator outputs \(z_i(t)\)
  \[
  z_i(t) = \int_0^T r(t)s_i(t)dt \quad i = 1, 2, \ldots, M
  \]

- Choose the waveform \(s_i(t)\) that matches best or the largest correlation with \(r(t)\)
Correlator receiver with reference signals \( \{\Psi_i(t)\} \)

- Recall: any signal set \( \{s_i(t)\} \) \( 1 \leq i \leq N \) can be expressed in terms of some basis functions
  \[ \{\Psi_i(t)\} \quad 1 \leq i \leq N \quad \text{where} \quad N \leq M \]

- the set of basis function \( \{\Psi_i(t)\} \) form reference signals
- choose the signal best match of the coefficients \( a_{ij} \) with the set of output \( \{z_i(T)\} \)
Detection of Signal in Gaussian Noise (7)

- Binary correlation receiver
Detection of Signal in Gaussian Noise (8)

Binary Decision Threshold

\[ \gamma_0 = \frac{a_1 + a_2}{2} = 0 \]
Coherent Detection of PSK

Consider: binary PSK (BPSK)

- antipodal

\[
\begin{align*}
    s_1(t) &= \sqrt{\frac{2E}{T}} \cos(\omega_0 t + \Phi) \\
    s_2(t) &= \sqrt{\frac{2E}{T}} \cos(\omega_0 t + \Phi + \pi) = -\sqrt{\frac{2E}{T}} \cos(\omega_0 t + \Phi)
\end{align*}
\]

where \(E\) : signal energy symbol, \(T\) : symbol duration

- orthonormal basis function

\[
\Psi_1(t) = \sqrt{\frac{2}{T}} \cos(\omega_0 t) \quad 0 \leq t \leq T
\]

→ only a single basis function is needed

\[
\begin{align*}
    s_1(t) &= a_{11} \Psi_1(t) = \sqrt{E} \Psi_1(t) \\
    s_2(t) &= a_{21} \Psi_1(t) = -\sqrt{E} \Psi_1(t)
\end{align*}
\]
Coherent Detection of PSK (cont)

Assume: $s_1(t)$ was transmitted, the expected value of the product integrator with reference signal $\Psi_1(t)$

$$E\{z_1|s_1\} = E \left\{ \int_0^T (\sqrt{E}\Psi_1^2(t) + n(t)\Psi_1(t))dt \right\} = \sqrt{E}$$

$$E\{z_2|s_1\} = E \left\{ \int_0^T (-\sqrt{E}\Psi_1^2(t) + n(t)\Psi_1(t))dt \right\} = -\sqrt{E}$$

Decision stage
- choose the signal with the largest value of $z_i(t) \rightarrow s_1(t)$
Sampled Matched Filter

Recall: impulse response $h(t)$ of a filter matched to $s(t)$

$$h(t) = \begin{cases} s(T - t) & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases}$$

How to implement a matched filter using digital hardware

$$r(k) = \sum_{n=0}^{N-1} r(k - n)c_i(n)$$
Coherent Detection (4)

Sampled Matched Filter

Coefficient or weight \( c_i(n) \) approximate an matched filter
\( (n = 0, 1, \ldots, N - 1), \quad N : \# \text{ of samples / symbol} \)
\[ c_i(n) = s_i[(N - 1) - n] = s_i(3 - n) \quad \text{if} \quad N = 4 \]

the output at a time corresponding to \( k \)th sample

\[
z_i(k) = \sum_{n=0}^{N-1} r(k - n)c_i(n) \quad k = 0, 1, \ldots, \text{modulo} - N
\]

Assume : BPSK transmission \((s_1(t), s_2(t))\)
\(s_1(t)\) was transmitted noise neglected
\(N = 4\)

\( c_i = \{s_i(3), s_i(2), s_i(1), s_i(0)\} \)

\[
z_i(k) = r(k)c_i(0) + r(k - 1)c_i(1) + r(k - 2)c_i(2) + r(k - 3)c_i(3)
\]

If \( k = 3 \),

\[
z_i(3) = r(3)c_i(0) + r(2)c_i(1) + r(1)c_i(2) + r(0)c_i(3)
\]
Sampled Matched Filter (cont)

\[ z_i(k) \quad i = 1, 2 \] outputs are examined at each value of \( k = N - 1 \)
Coherent Detection of Multiple Phase-Shift Keying (MPSK)

Signal space and decision regions for a QPSK system
Coherent Detection of MPSK (cont)

For typical coherent MPSK systems

\[ s_i(t) = \sqrt{\frac{2E}{T}} \cos \left( \omega_0 t - \frac{2\pi i}{M} \right) \quad \left\{ \begin{array}{l} 0 \leq t \leq T \\ i = 1, 2, \ldots, M \end{array} \right. \]

where \( T = 2T_b : \text{symbol duration} \)
\( E = 2E_b : \text{received energy} \)

Orthonormal signal space

\[ \Psi_1(t) = \sqrt{\frac{2}{T}} \cos \omega_0 t \quad \Psi_2(t) = \sqrt{\frac{2}{T}} \sin \omega_0 t \]

\[ s_i = a_{i1} \Psi_1(t) + a_{i2} \Psi_2(t) \quad \left\{ \begin{array}{l} 0 \leq t \leq T \\ i = 1, 2, \ldots, M \end{array} \right. \]

\[ = \sqrt{E} \cos \left( \frac{2\pi i}{M} \right) \Psi_1(t) + \sqrt{E} \sin \left( \frac{2\pi i}{M} \right) \Psi_2(t) \]
Coherent Detection (8)

- Coherent Detection of MPSK (cont)
- Demodulator for MPSK signal

\[ r(t) = \sqrt{\frac{2E}{T}} (\cos \Phi_i \cos \omega_0 t + \sin \Phi_i \sin \omega_0 t) + n(t) \]

\[ \{ \begin{array}{l} 
0 \leq t \leq T \\
 i = 1, 2, \ldots, M 
\end{array} \]
Coherent Detection of FSK

Typical set of FSK signal waveforms

\[ s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t + \Phi) \quad \begin{cases} \text{for} & 0 \leq t \leq T \\ \text{with} & i = 1, 2, \ldots, M \end{cases} \]

orthonormal basis function

\[ \Psi_j(t) = \sqrt{\frac{2}{T}} \cos \omega_j t \quad j = 1, 2, \ldots, N \]

\[ a_{ij} = \int_{0}^{T} s_i(t)\Psi_j(t)dt = \begin{cases} \sqrt{E} & \text{for} \ i = j \\ 0 & \text{otherwise} \end{cases} \]

distance between any two prototype signal vectors \( s_i \) and \( s_j \) is constant

\[ d(s_i, s_j) = \| s_i - s_j \| = \sqrt{2E} \quad \text{for} \ i \neq j \]
Coherent Detection of FSK (cont)

Partitioning the signal space for 3-ary FSK signal
Detection of Differential PSK

- Recall
  - Coherent: require a reference in phase with the received carrier
  - Noncoherent: do not require a reference in phase with the received carrier

- Differential Encoding: information is carried by the difference in phase between two successive waveforms

<table>
<thead>
<tr>
<th>BPSK</th>
<th>“1”: 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>“0”: ( \pi )</td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\text{s(t+ T)} & \quad \text{위상변화} & \quad \text{X} & \quad \text{s(t)} \quad \text{위상변화180°} \\
\text{“1”} & \quad \text{s(t)} & \quad \text{위상변화} & \quad \text{X} & \quad \text{s(t)} \quad \text{위상변화180°}
\end{align*}
\]
Noncoherent Detection (2)

Detection of Differential PSK (cont)

Differentially coherent detection

- the carrier phase of the previous signaling is used as a phase reference

\[ s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_0 t + \theta_i(t)) \quad \begin{cases} \quad 0 \leq t \leq T \\ i = 1,2, \ldots, M \end{cases} \]

- Assume: \( \alpha \) varies slowly relative to two period time (2T) phase difference between two successive waveforms

\[ \theta_j(T_1) \& \theta_k(T_2) \]

\[ [\theta_k(T_2) + \alpha] - [\theta_j(T_1) + \alpha] = \theta_k(T_2) - \theta_j(T_1) = \theta_i(T_2) \]

- Observation: phase difference is independent of \( \alpha \)

<table>
<thead>
<tr>
<th>PSK</th>
<th>DPSK</th>
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<tbody>
<tr>
<td>Compare the received signal with a clean reference</td>
<td>Error propagation to adjacent symbol times Two noisy signals are compared with each other (3dB worse) Reduce system complexity</td>
</tr>
</tbody>
</table>
Noncoherent Detection (3)

**Binary Differential PSK Example**

Differential Encoding: \( c(k) = c(k - 1) \oplus m(k) \)

or \( c(k) = \overline{c(k - 1)} \oplus m(k) \)

<table>
<thead>
<tr>
<th>Sample index, ( k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Information message, ( m(k) )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Differentially encoded message (first bit arbitrary), ( c(k) )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Corresponding phase shift, ( \theta(k) )</td>
<td>( \pi )</td>
<td>( \pi )</td>
<td>( \pi )</td>
<td>0</td>
<td>0</td>
<td>( \pi )</td>
<td>( \pi )</td>
<td>( \pi )</td>
<td>0</td>
<td>( \pi )</td>
<td>( \pi )</td>
</tr>
</tbody>
</table>

\( \{ m(k) = c(k - 1) \rightarrow c(k) = 1 \)  
\( \{ m(k) \neq c(k - 1) \rightarrow c(k) = 0 \)
Binary Differential PSK Example (cont)

- Differential coherent detection
  - reference: delayed version of the received signal
  - coherent detector: match a received symbol with the prior symbol and look for a correlation or an anticorrelation (180° out of phase)
Noncoherent Detection of FSK

implemented as an "energy detector" without exploiting phase measurement

Consider: $T_x$ signal

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos (2\pi f_i t) \quad 0 \leq t \leq T$$

$$\downarrow$$

$$\oplus \leftarrow \text{AWGN}$$

$$\downarrow$$

$$r(t) = \sqrt{\frac{2E}{T}} \cos (2\pi f_i t + \Theta) + n(t)$$

$$= \sqrt{\frac{2E}{T}} \left[ \cos \Theta \cos (2\pi f_i t) - \sin \Theta \sin (2\pi f_i t) \right] + n(t)$$
Noncoherent Detection of FSK (cont)

Assume: no noise

\[ r(t) = \sqrt{\frac{2E}{T}} [\cos \theta \cos(2\pi f_i t) - \sin \theta \sin(2\pi f_i t)] \]

BFSK \((f_1 \& f_2)\)

- The incoming signal will partially correlate with the \(\cos \omega_i t\) reference and partially correlate with the \(\sin \omega_i t\) reference → require an I & Q branch for each signal
Noncoherent Detection (7)

Noncoherent Detection of FSK (cont)

Noncoherent detection using envelope detector
Noncoherent Detection (8)

Required Tone Spacing for Noncoherent Orthogonal FSK Signaling

Requirement for orthogonality between tones in an FSK signaling set
→ any pair if tones in the set must have a frequency separation that is a multiple of 1/T Hz

\[ s_i(t) = (\cos 2\pi f_i t) \text{rect} \left( \frac{t}{T} \right) \quad \mathcal{F} \quad f\{s_i(t)\} = Tsinc(f - f_i)T \]

BFSK \((f_1 \text{ & } f_2)\)
Noncoherent Detection (9)

Required Tone Spacing for Noncoherent OFSK Signaling (cont)

Minimum tone spacing & bandwidth

- minimum tone spacing for noncoherent detection : $\frac{1}{T}$ Hz
- bandwidth of noncoherently detected orthogonal MFSK : $\frac{M}{T}$ Hz
- cf) minimum tone spacing for coherent detection : $\frac{1}{2T}$ Hz
Complex Envelope

- bandpass waveform \( s(t) \): complex notation

\[
s(t) = \text{Re}\{g(t)e^{j\omega_0 t}\}
\]

\[
\begin{align*}
  g(t) &: \text{ complex envelope,} \\
  e^{j\omega_0 t} &: \text{ complex carrier wave}
\end{align*}
\]

\[
g(t) = x(t) + jy(t) = |g(t)|e^{j\theta(t)} = R(t)e^{j\theta(t)}
\]

\[
\begin{align*}
  R(t) &: \text{ magnitude of the complex envelope} \\
  e^{j\theta(t)} &: \text{ phase of the complex envelope}
\end{align*}
\]

\[
\begin{align*}
  R(t) &= |g(t)| = \sqrt{x^2(t) + y^2(t)} \\
  \theta(t) &= \tan^{-1}\frac{y(t)}{x(t)}
\end{align*}
\]

\[
s(t) = \text{Re}\{[x(t) + jy(t)][\cos\omega_0 t + j\sin\omega_0 t]\}
\]

\[
= x(t)\cos\omega_0 t - y(t)\sin\omega_0 t
\]
Quadrature Implementation of a Modulator

Consider: an example of baseband waveform

\[ g(t) = g_k \quad k = 1, 2, \ldots \ldots \]
\[ g_k = x_k + jy_k = 0.707A + j0.707A \]

Quadrature type modulator

\[ s(t) = Re\{g_k e^{j\omega_0 t}\} \]
\[ = Re\{(x_k + jy_k)(\cos\omega_0 t + jsin\omega_0 t)\} \]
\[ = x_k \cos\omega_0 t - y_k \sin\omega_0 t \]
\[ = 0.707A \cos\omega_0 t - 0.707A \sin\omega_0 t \]
\[ = A \cos(\omega_0 t + \frac{\pi}{4}) \]

Modulation process can be described as multiplying the complex envelope by \(e^{j\omega_0 t}\) and then transmitting real part.
D8PSK Modulator Example

Differential 8-PSK (D8PSK) Modulator for tone $k=2$

$$s(t) = -0.707\cos\omega_0 t + 0.707\sin\omega_0 t = \sin\left(\omega_0 t - \frac{\pi}{4}\right) = \cos\left(\omega_0 t + \frac{5\pi}{4}\right)$$
D8PSK Demodulator Example

\[ A' \land B' \Rightarrow \Phi_k \Rightarrow \Delta \Phi_k = \Phi_k - \Phi_{k-1} \]
Probability of Bit Error for Coherently Detected BPSK ($P_B = P_E$)

- Assume: signals ($s_1(t)$ & $s_2(t)$) are equally likely
- Received signal $r(t) = s_i(t) + n(t)$

\[
\begin{align*}
    s_1(t) &= \sqrt{\frac{2E}{T}} \cos \omega_0 t \quad \text{"antipodal signal"} \\
    s_2(t) &= -\sqrt{\frac{2E}{T}} \cos \omega_0 t
\end{align*}
\]

- Characterized in a one-dimensional signal space

\[
\begin{align*}
    \begin{cases} 
    s_1(t) = \sqrt{E} \Psi_1(t) \\
    s_2(t) = -\sqrt{E} \Psi_1(t)
    \end{cases}
\end{align*}
\]

- Decision rule

\[
\begin{align*}
    \begin{cases} 
    s_1(t) & \text{if } z(T) > r_0 = 0 \\
    s_2(t) & \text{otherwise}
    \end{cases}
\end{align*}
\]
Error Performance for Binary Systems (2)

- Probability of Bit Error for Coherently Detected BPSK (cont)
  - \( P_B \) for binary minimum error detector
    \[
    P_B = \int_{\frac{a_1-a_2}{2\sigma_0}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{u^2}{2} \right) du = Q \left( \frac{a_1-a_2}{2\sigma_0} \right)
    \]
    for BPSK \( a_1 = \sqrt{E_b}, \quad a_2 = -\sqrt{E_b}, \quad \sigma_0^2 = \frac{N_0}{2} \)
    \[
    P_B = \int_{\frac{2E_b}{\sqrt{N_0}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{u^2}{2} \right) du = Q \left( \frac{2E_b}{\sqrt{N_0}} \right)
    \]
  - Observation
    - this result for bandpass antipodal BPSK is the same as the result that wave developed for the matched filter detection of baseband antipodal signal
Probability of Bit Error for Coherently Detected, Differentially Encoded Binary PSK

\[ P_B = 2Q \left( 2 \sqrt{\frac{2E_b}{N_0}} \right) \left[ 1 - Q \left( 2 \sqrt{\frac{2E_b}{N_0}} \right) \right] \]
**Error Performance for Binary Systems (4)**

- **Probability of Bit Error for Coherently Detected Binary Orthogonal FSK**
  - Recall: \( P_B = Q \left( \sqrt{\frac{E_b(1-\rho)}{N_0}} \right) \) \hspace{1cm} where \( \rho = \cos \theta \)
  - BFSK: \( \theta = \frac{\pi}{2} \Rightarrow \rho = 0 \) \hspace{1cm} \therefore P_B = Q \left( \sqrt{\frac{E_b}{N_0}} \right)
  - Observation: 3dB worse than BPSK

- **Probability of Bit Error for Noncoherently Detected Binary Orthogonal FSK**
  - Observation: require approximately 1dB more \( E_b/N_0 \) than that for coherent FSK but easier to implement
  - \( P_B = \frac{1}{2} \exp \left( -\frac{E_b}{2N_0} \right) \)
## Error Performance for Binary Systems (5)

- **Probability of Bit Error for Binary DPSK**
  - Observation: require approximately 1dB more $E_b/N_0$ than BPSK ($P_B \leq 10^{-4}$) but easier to implement
  
  $P_B = \frac{1}{2} \exp \left(-\frac{E_b}{N_0}\right)$ for DPSK detector

- **Comparison of Bit Error Performance for Various Modulation Types**

<table>
<thead>
<tr>
<th>Modulation</th>
<th>$P_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PSK (coherent)</td>
<td>$Q\left(\sqrt{2E_b/N_0}\right)$</td>
</tr>
<tr>
<td>DPSK (differentially coherent)</td>
<td>$\exp(-E_b/N_0)/2$</td>
</tr>
<tr>
<td>Orthogonal FSK (coherent)</td>
<td>$Q\left(\sqrt{E_b/N_0}\right)$</td>
</tr>
<tr>
<td>Orthogonal FSK (noncoherent)</td>
<td>$\exp(-E_b/2N_0)/2$</td>
</tr>
</tbody>
</table>
M-ary Signaling and Performance (1)

Ideal Probability of Bit Error Performance

![Typical $P_B$ versus $E_b/N_0$ curve]

- Ideal curve
- Shannon limit: -1.6 dB
- Direction of movement for $P_B$ improvement
M-ary Signaling

- Coherently detected MPSK signaling over a AWGN
- Observation: as increase, error performance is degraded but require bandwidth is decreased
- Coherently detected orthogonal M-ary signaling over a AWGN
M-ary Signaling and Performance (3)

- M-ary Signaling (cont)
  - As $k$ increases
    - improved error performance
    - bandwidth increases
  - Error performance improvement can be achieved at the expense of bandwidth
  - Error performance vs bandwidth performance
    - trade-off
Vectorial View of MPSK Signaling

- MPSK signal set for M=2, 4, 8, 16
- All phasors have the same length for any of the M-ary cases (fixed $E_s/N_0$)

As M is increased → crowded more signal vector
→ increase error performance
→ improve bandwidth performance at the expense of increasing $E_b/N_0$
If we increase $E_b/N_0$ so that the $P_B$ is not degraded
BPSK and QPSK Have the Same Bit Error Probability

General relationship between $E_b/N_0$ and $S/N$ for binary transmission

\[
\frac{E_b}{N_0} = \frac{S}{N} \left(\frac{W}{R}\right)
\]

where

$S = \text{average signal power}$

$R = \text{bit rate}$

Observation

- each of the orthogonal BPSK channels, and hence the composite QPSK signal, is characterized by the same $E_b/N_0$, and hence the same $P_B$ performance as a BPSK signal
- $P_B(BPSK) = P_B(QPSK)$, $P_E(BPSK) \neq P_E(QPSK)$
Vectorial view of MFSK Signaling

MFSK signal sets for $M = 2, 3$
Vectorial view of MFSK Signaling (cont)

- Observation
  - MFSK signal space is characterized by M mutually perpendicular axes

![Graph showing symbol error probability as a function of SNR for different values of M (2, 3, 4, 8, 16, 32, and 1024).]
M-ary Signaling and Performance (8)

Vectorial view of MFSK Signaling (cont)

- Mapping from $P_E$ vs SNR to $P_E$ vs $\frac{E_b}{N_0}$

\[
\frac{E_b}{N_0} = \frac{S}{N} \left( \frac{W}{R} \right) \quad \text{where } W : \text{detection bandwidth}
\]

Since $R = \frac{k}{T} = \frac{\log_2 M}{T}$ where $T : \text{symbol duration}$

\[
\therefore \frac{E_b}{N_0} = \frac{S}{N} \left( \frac{WT}{\log_2 M} \right) = \frac{S}{N} \left( \frac{WT}{k} \right) \approx \frac{S}{N} \left( \frac{1}{k} \right) \quad \{ \text{typical} \}
\]

- Expressed in SNR [dB]

SNR [dB]

\[
= 10 \log \left( \frac{P_{\text{signal}}}{P_{\text{noise}}} \right)
\]

\[
= 10 \log \left( \frac{A_{\text{signal}}}{A_{\text{noise}}} \right)^2
\]
M-ary Signaling and Performance (9)

- Vectorial view of MFSK Signaling (cont)
  - Expressed in $E_b/N_0$ [dB]
    
    $$
    \frac{E_b}{N_0} [dB] = \frac{S}{N} [dB] - 10 \log k
    $$

    - "improved with increasing M", "degraded with increasing M"
  - When we use orthogonal signaling with symbols that contain more bits(M) we need more power (more SNR) but the requirement per bit($E_b/N_0$) is reduced
Symbol Error Performance for M-ary Systems (1)

- **Probability of Symbol Error for MPSK**
  - $P_E$ for coherently detected MPSK
    \[
    P_E(M) \approx 2Q \left( \sqrt{\frac{2E_S}{N_0}} \sin \frac{\pi}{M} \right)
    \]
    where
    \[
    \begin{align*}
    E_S &= E_b (\log_2 M) \\
    M &= 2^k
    \end{align*}
    \]
  - $P_E$ for differentially coherent detection of M-ary DPSK
    \[
    P_E(M) \approx 2Q \left( \sqrt{\frac{2E_S}{N_0}} \sin \frac{\pi}{\sqrt{2M}} \right)
    \]
Symbol Error Performance for M-ary Systems (2)

- Probability of Symbol Error for MFSK
  - $P_E$ for coherently detected MFSK
    \[
P_E(M) \leq (M - 1)Q \left( \frac{E_s}{\sqrt{N_0}} \right)
    \]
  - $P_E$ for noncoherently detected MFSK
    \[
    P_E(M) = \frac{1}{M} \exp \left( \frac{E_s}{N_0} \sum_{j=2}^{M} (-1)^j \binom{M}{j} \exp \left( \frac{E_s}{jN_0} \right) \right)
    \]
  - M개 중에 j개의 오류가 있을 확률
    \[
    \binom{M}{j} = \frac{M!}{j!(M-j)!}
    \]
Symbol Error Performance for M-ary Systems (3)

- Bit Error Probability versus Symbol Error Probability for Orthogonal Signals
  - Example of $P_S$ vs $P_E$
    - $\frac{P_B}{P_E} = \frac{2^{k-1}}{2^k - 1} = \frac{M/2}{M-1}$ for non-binary signaling $P_B < P_E$
Bit Error Probability versus Symbol Error Probability for Multiple Phase Signaling

Assume: symbol (011) is transmitted if an error occurs, (010) or (100) is more likely

Gray code: adjacent symbols differ in only one bit position

\[ P_B \approx \frac{P_E}{\log_2 M} \quad (for \quad P_E \ll 1) \]

- BPSK: \( P_B = P_E \)
- QPSK: \( P_B = \frac{P_E}{2} \rightarrow P_E = 2P_B \)
Thank you for giving your attention!